Characterization of multi-mode linear optical network
Supplementary materials

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S.I Quantum correlations

In this section, we recall the proof of Eq. (2) of the main text. First, we calculate the states at the outputs i and j when two photons, respectively in the state $|\phi\rangle$ e $|\psi\rangle$, are injected in the input modes h and k. We first choose an orthonormal basis to express the states on the two input modes:

$$|\phi\rangle_h = \sum_l \alpha_l a^\dagger_{lh} |0\rangle \quad \sum_l |\alpha_l|^2 = 1 \qquad |\psi\rangle_k = \sum_m \beta_m a^\dagger_{mk} |0\rangle \quad \sum_m |\beta_m|^2 = 1 \tag{S1}$$

Then, the two-photon states is expanded as:

$$|\phi\rangle_h \otimes |\psi\rangle_k = \sum_{l,m} \alpha_l \beta_m a^\dagger_{lh} a^\dagger_{mk} |0\rangle \tag{S2}$$

The state after the unitary evolution is:

$$\begin{aligned}
\tilde{U} |\phi\rangle_h \otimes |\psi\rangle_k &= \sum_{l,m} \alpha_l \beta_m \hat{U} a^\dagger_{lh} \hat{U}^\dagger \hat{U} a^\dagger_{mk} \hat{U}^\dagger |0\rangle \\
&= \sum_{l,m} \alpha_l \beta_m (U_{ih} a^\dagger_{li} + U_{jh} a^\dagger_{lj})(U_{ik} a^\dagger_{mi} + U_{jk} a^\dagger_{mj}) |0\rangle \\
&= \sum_{l,m} \alpha_l \beta_m (U_{ih} U_{jk} a^\dagger_{li} a^\dagger_{mj} + U_{jh} U_{ik} a^\dagger_{lj} a^\dagger_{mi} + U_{ih} U_{ik} a^\dagger_{li} a^\dagger_{mi} + U_{jh} U_{jk} a^\dagger_{lj} a^\dagger_{mj}) |0\rangle \\
&= \sum_{l,m} \underbrace{(U_{ih} U_{jk} \alpha_l \beta_m + U_{jh} U_{ik} \alpha_m \beta_l) a^\dagger_{li} a^\dagger_{mj}}_{\text{non-collisional term}} + \underbrace{\alpha_l \beta_m (U_{ih} U_{ik} a^\dagger_{il} a^\dagger_{im} + U_{jh} U_{jk} a^\dagger_{jl} a^\dagger_{jm})}_{\text{colisional term}} |0\rangle
\end{aligned} \tag{S3}$$

Since we are interested in the probability to measure two photons in two different output ports $i \neq j$, we calculate the probability that the output state is in the non-collisional term

$$\begin{aligned}
P^{hk}_{ij} &= \sum_{l,m} (U_{ih} U_{jk} \alpha_l \beta_m + U_{jh} U_{ik} \alpha_m \beta_l)(U_{ih} U_{jk} \alpha_l \beta_m + U_{jh} U_{ik} \alpha_m \beta_l)^* \\
&= \sum_{l,m} |U_{ih}|^2 |U_{jk}|^2 |\alpha_i|^2 |\beta_j|^2 + |U_{jh}|^2 |U_{ik}|^2 |\alpha_j|^2 |\beta_i|^2 + U_{ih} U_{jk} U^*_{jh} U^*_{ik} \alpha_l \beta_m \alpha^*_m \beta^*_l + U_{jh} U_{ik} U^*_{ih} U^*_{jk} \alpha_m \beta_l \alpha^*_l \beta^*_m \\
&= |U_{ih}|^2 |U_{jk}|^2 + |U_{jh}|^2 |U_{ik}|^2 + (U_{ih} U_{jk} U^*_{jh} U^*_{ik} + U_{jh} U_{ik} U^*_{ih} U^*_{jk}) |\langle \phi | \psi \rangle|^2
\end{aligned} \tag{S4}$$

In this expression, we observe that the indistinguishable photons scenario is obtained for $|\langle \phi | \psi \rangle|^2 = 1$, while the distinguishable particle case corresponds to $|\langle \phi | \psi \rangle|^2 = 0$. Using the definition of the HOM visibility, we find Eq. (2) of the main text.

$$\mathcal{V}^{hk}_{ij} = 1 - \frac{(P^{hk}_{ij})^{\mathrm{I}}}{(P^{hk}_{ij})^{\mathrm{D}}} = -\frac{U_{ih} U_{jk} U^*_{jh} U^*_{ik} + U_{jh} U_{ik} U^*_{ih} U^*_{jk}}{|U_{ih}|^2 |U_{jk}|^2 + |U_{jh}|^2 |U_{ik}|^2} = -\frac{2\tau_{jk}\tau_{ih}\tau_{ik}\tau_{jh}}{\tau^2_{jk}\tau^2_{ih} + \tau^2_{ik}\tau^2_{jh}} \cos(\phi_{jk} + \phi_{ih} - \phi_{ik} - \phi_{jh}) \tag{S5}$$

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When residual distinguishability is present the measured visibility follows the following equation:

$$\mathcal{V}_{ij}^{hk} = -\frac{2\tau_{jk}\tau_{ih}\tau_{ik}\tau_{jh}}{\tau_{jk}^2\tau_{ih}^2 + \tau_{ik}^2\tau_{jh}^2}\cos\left(\phi_{jk} + \phi_{ih} - \phi_{ik} - \phi_{jh}\right)\left|\langle\phi|\psi\rangle\right|^2 \tag{S6}$$

If this effect is not properly taken into account, this can lead to errors in unitary reconstruction algorithms based on such quantity.

Finally, we observe that is possible to obtain a quantity analogous to the normalized correlation of Eq. (S15), starting from visibility measurements. This procedure requires the measurements of the bunching probabilities, that is, the probability that two photons exit from the same output mode. Indeed, using this information it is possible to can reconstruct the following quantity

$$T_{ij}^{hk} = \frac{(P_{ij}^{hk})^{\mathrm{I}} - (P_{ij}^{hk})^{\mathrm{D}}}{\sqrt{[(P_{ii}^{hk})^{\mathrm{I}} - (P_{ii}^{hk})^{\mathrm{D}}][(P_{jj}^{hk})^{\mathrm{I}} - (P_{jj}^{hk})^{\mathrm{D}}]}} = \frac{(U_{ih}U_{ik}^*U_{jh}^*U_{jk} + U_{ih}^*U_{ik}U_{jh}U_{jk}^*)}{2|U_{ih}||U_{ik}||U_{jh}||U_{jk}|} = \cos(\phi_{ih} - \phi_{ik} - \phi_{jh} + \phi_{jk}) \tag{S7}$$

S.II Classical correlations

In this section, we show that second-order classical correlation measurements are described by an expression which is equivalent to two-photon Hong-Ou-Mandel visibilities. Starting from Figure S1, we consider two laser beams at the input of the interferometer. The input fields are described as:

$$\tilde{E}_h = \tilde{E}_1 e^{i\varphi_1(t)} \tag{S8}$$
$$\tilde{E}_k = \tilde{E}_2 e^{i\varphi_2(t)} \tag{S9}$$

where $\phi_1(t)$ and $\phi_2(t)$ are the phases introduced via propagation in optical fibers, and by all the other possible optical delays in the apparatus. After propagation through the interferometer, the electric fields on the output modes read:

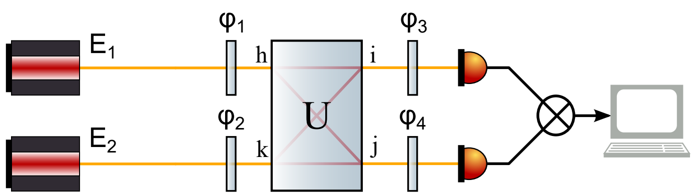

FIG. S1. Conceptual scheme of the apparatus for the reconstruction of matrix phases via second-order correlations with classical light.

$$\begin{aligned}\tilde{E}_i &= U_{ih}\tilde{E}_h + U_{ik}\tilde{E}_k\\ \tilde{E}_j &= U_{jh}\tilde{E}_h + U_{jk}\tilde{E}_k\end{aligned} \tag{S10}$$

Then, the output intensity of light is measured via two photodiodes:

$$\begin{aligned}I_i &= \tilde{E}_i\tilde{E}_i^* = I_1\tau_{ih}^2 + I_2\tau_{ik}^2 + \tilde{E}_1\tilde{E}_2^*U_{ih}U_{ik}^* + \tilde{E}_1^*\tilde{E}_2U_{ih}^*U_{ik}\\ I_j &= \tilde{E}_j\tilde{E}_j^* = I_1\tau_{jh}^2 + I_2\tau_{jk}^2 + \tilde{E}_1\tilde{E}_2^*U_{jh}U_{jk}^* + \tilde{E}_1^*\tilde{E}_2U_{jh}^*U_{jk}\end{aligned} \tag{S11}$$

If we suppose that the intensity of the input lasers I_1 and I_2 are constant and that $\langle\tilde{E}_1\tilde{E}_2^*\rangle = 0$ (that in our scenario is equivalent to $\langle e^{\imath(\varphi_1-\varphi_1}\rangle = 0$), we can calculate the residual as:

$$\begin{aligned}I_i - \langle I_i\rangle &= \tilde{E}_1\tilde{E}_2^*U_{ih}U_{ik}^* + \tilde{E}_1^*\tilde{E}_2U_{ih}^*U_{ik}\\ I_j - \langle I_j\rangle &= \tilde{E}_1\tilde{E}_2^*U_{jh}U_{jk}^* + \tilde{E}_1^*\tilde{E}_2U_{jh}^*U_{jk}\end{aligned} \tag{S12}$$

At this point we can define the cross-correlation σ_{ij}^{hk} between the output modes (i,j) when the two beams enter from modes (h,k), and the self-correlation σ_{ii}^{hk} of the intensity fluctuation a:

$$\begin{aligned}\sigma_{ij}^{hk} &= \left\langle(I_i - \langle I_i\rangle)(I_j - \langle I_j\rangle)\right\rangle\\ \sigma_{ii}^{hk} &= \left\langle(I_i - \langle I_i\rangle)^2\right\rangle\end{aligned} \tag{S13}$$

Let us know define a parameter $\gamma = \langle\tilde{E}_1\tilde{E}_2^*\tilde{E}_1^*\tilde{E}_2\rangle/(I_1I_2)$, related to the first order correlation functions of the two beams that it is in turn related to visibility of the interference fringes. In addition, we assume that the fields satisfy the following hypothesis

$\langle(\tilde{E}_1\tilde{E}_2^*)^2\rangle = 0$, which is equivalent to the condition $\langle e^{\imath 2(\varphi_1-\varphi_1}\rangle = 0$. Under these assumptions, the cross-correlations can be calculated as:

$$\begin{aligned}\sigma_{ij}^{hk} &= \gamma I_1 I_2 (U_{ih}U_{ik}^* U_{jh}^* U_{jk} + U_{ih}^* U_{ik} U_{jh} U_{jk}^*)\\ \sigma_{ii}^{hk} &= 2\gamma I_1 I_2 |U_{ih}|^2 |U_{ik}|^2\\ \sigma_{jj}^{hk} &= 2\gamma I_1 I_2 |U_{jh}|^2 |U_{jk}|^2\end{aligned} \tag{S14}$$

Finally, by defining the normalized cross-correlation C_{ij}^{hk} we obtain Eq. (19) of the main text:

$$C_{ij}^{hk} = \frac{\sigma_{ij}^{hk}}{\sqrt{\sigma_{ii}^{hk}\sigma_{jj}^{hk}}} = \frac{(U_{ih}U_{ik}^* U_{jh}^* U_{jk} + U_{ih}^* U_{ik} U_{jh} U_{jk}^*)}{2|U_{ih}||U_{ik}||U_{jh}||U_{jk}|} = \cos(\phi_{ih} - \phi_{ik} - \phi_{jh} + \phi_{jk}) \tag{S15}$$

We note that these quantities do not depend on the visibility γ nor from the moduli of the unitary matrix. We obtained something equivalent in the quantum correlation section but with the need for single-photon number resolving detection to evaluate the bunching probability.

S.III Chip fabrication

The photonic chip was fabricated in a borosilicate glass substrate (EagleXG, from Corning Inc., USA) by femtosecond laser direct writing. In detail, irradiation with focused femtosecond laser pulses causes a permanent refractive index increase in the glass, localized in the focal region (see Fig. S2); translation of the substrate with respect to the laser beam allows for the direct inscription of waveguides along the desired paths. A Yb-based cavity-dumped oscillator was adopted as a laser source, emitting ultrashort pulses with 1030 nm wavelength and 300 fs duration at the repetition rate of 1 MHz. For waveguide inscription, laser pulses of 250 nJ energy were focused 30 μm below the substrate surface by a 0.6 NA microscope objective, while the substrate was translated at a constant speed of 30 mm/s. These parameters enabled the fabrication of single-mode waveguides operating at the wavelength of 785 nm, with a $1/e^2$ mode size of $7.2\mu m \times 8.4\mu m$ and propagation losses (for vertical polarization) lower than 0.8 dB/cm. The depth of the waveguides was chosen to allow the control of the optical phase via thermo-optic phase shifters. The thermal shifters consist of metallic resistors, manufactured by the same femtosecond laser through the ablation of a thin and uniform gold layer ($\sim$ 60-nm thickness) deposited on the chip surface. Each resistor has a width of 100 μm and a length between 5 and 7 mm (parallel to the waveguide direction), which gives resistance values in the range 60-100 Ω. Electrical connections are provided via standard pins, directly glued on the electrical circuit terminations.

FIG. S2. Typical cross-section of an optical waveguide inscribed with a single irradiation step of femtosecond laser in the borosilicate glass substrate, imaged with an optical microscope. The parameters of the laser beam and the focusing conditions are described in the text. The femtosecond laser is impinging from the top in the figure, and the scale bar corresponds to 10 μm. The refractive index modification assumes a drop-like shape with complex features; however, the waveguide supports a single optical mode at the wavelength of interest.